Lemma 3.7. *Let T be a Hausdorff topological space and C_1, C_2 disjoint compact subsets of T. Then, there are disjoint open subsets U_1, U_2 of T such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. In particular, if T is compact, then it is normal.*

Proof. We first show a weaker statement: Let C be a compact subset of T and $p \notin C$. Then there exist disjoint open sets U and V such that $p \in U$ and $C \subseteq V$. Since T is Hausdorff, for each point $q \in C$ there exist disjoint open sets U_q and V_q such that $p \in U_q$ and $q \in V_q$. The family of sets $\{V_q\}_{q\in C}$ defines an open covering of C. Since C is compact there is a finite subset $S \subseteq C$ such that the family $\{V_q\}_{q\in S}$ already covers C. Define $U := \bigcap_{q\in S} U_q$ and $V := \bigcup_{q\in S} V_q$. These are open sets with the desired properties.

We proceed to the prove the first statement of the lemma. By the previous demonstration, for each point $p \in C_1$ there are disjoint open sets U_p and V_p such that $p \in U_p$ and $C_2 \subseteq V_p$. The family of sets $\{U_p\}_{p\in C_1}$ defines an open covering of C_1. Since C_1 is compact there is a finite subset $S \subseteq C_1$ such that the family $\{U_p\}_{p\in S}$ already covers C_1. Define $U_1 := \bigcup_{p\in S} U_p$ and $U_2 := \bigcap_{p\in S} V_p$.

For the second statement of the lemma observe that if T is compact, then every closed subset is compact. □

Lemma 3.8 (Urysohn's Lemma)**.** *Let T be a normal topological space and C_1, C_2 disjoint closed subsets of T. Then there exists a continuous function $f : T \to [0, 1]$ such that $f(C_1) = 0$ and $f(C_2) = 1$.*

Together with the above lemmas the Gelfand-Naimark theorem gives rise to a one-to-one correspondence between compact Hausdorff spaces and unital commutative C*-algebras.

Theorem 3.9. *The category of compact Hausdorff spaces is naturally equivalent to the category of unital commutative C*-algebras.*

Proof. **Exercise.** □

Before we proceed we need a few more results about C*-algebras.

Lemma 3.10. *Let T_1 be a compact Hausdorff space, T_2 be a Hausdorff space and $f : T_1 \to T_2$ a continuous bijective map. Then, f is a homeomorphism.*

Proposition 3.11. *Let A be a unital C*-algebra and $a \in A$ normal. Define B to be the unital C*-subalgebra of A generated by a. Then, B is commutative and the Gelfand transform $\hat{a}$ of a defines a homeomorphism onto its image, $\Gamma_B \to \sigma_B(a)$ which we denote by $\tilde{a}$.*

Proof. B consists of possibly infinite linear combinations of elements of the form $(a^*)^m a^n$ where $n, m \in \mathbb{N}_0$ (and $a^0 = (a^*)^0 = e$). In particular, B is commutative. Consider the Gelfand transform $\hat{a} : \Gamma_B \to \mathbb{C}$ of a in B. Suppose $\hat{a}(\phi) = \hat{a}(\psi)$ for $\phi, \psi \in \Gamma_B$. Then, $\phi(a) = \psi(a)$, but also

$$\phi(a^*) = \hat{a^*}(\phi) = \overline{\hat{a}(\phi)} = \overline{\hat{a}(\psi)} = \hat{a^*}(\psi) = \psi(a^*),$$

using Theorem 3.6. Thus, ϕ is equal to ψ on monomials $(a^*)^m a^n$ by multiplicativity and hence on all of B by linearity and continuity. This shows that $\hat{a}$ is injective. By Proposition 2.15 the image of $\hat{a}$ is $\sigma_B(a)$. Thus, $\hat{a}$ is a continuous bijective map $\hat{a} : \Gamma_B \to \sigma_B(a)$. With Lemma 3.10 it is even a homeomorphism. □

Proposition 3.12. *Let A be a unital C*-algebra and $a \in A$. Let B be a untial C*-subalgebra containing e and a. Then, $\sigma_B(a) = \sigma_A(a)$.*

Proof. It is clear that $\sigma_A(a) \subseteq \sigma_B(a)$. It remains to show that if $b := \lambda e - a$ for any $\lambda \in \mathbb{C}$ has an inverse in A then this inverse is also contained in B.

Assume first that a (and hence b) is normal. We show that b^{-1} is even contained in the unital C*-subalgebra C of B that is generated by b. Suppose that b^{-1} is not contained in C and hence $0 \in \sigma_C(b)$. Choose $m > \|b^{-1}\|$ and define a continuous function $f : \sigma_C(b) \to \mathbb{C}$ such that $f(0) = m$ and $|f(x)x| \leq 1$ for all $x \in \sigma_C(b)$. Using Theorem 3.6 and Proposition 3.11 there is a unique element $c \in C$ such that $\hat{c} = f \circ \tilde{b}$. Observe also that $\hat{b} = \mathrm{id} \circ \tilde{b}$, where $\mathrm{id} : \sigma_C(b) \to \mathbb{C}$ is the map $x \mapsto x$ and hence $\hat{cb} = (f \cdot \mathrm{id}) \circ \tilde{b}$. Using Theorem 3.6 we find

$$m \leq \|f\| = \|c\| = \|cbb^{-1}\| \leq \|cb\|\|b^{-1}\| = \|f \cdot \mathrm{id}\|\|b^{-1}\| \leq \|b^{-1}\|.$$

This contradicts $m > \|b^{-1}\|$. So $0 \notin \sigma_C(b)$ and $b^{-1} \in C$ as was to be demonstrated. This proves the proposition for a normal.

Consider now the general case. If b is not invertible in B then by Lemma 1.8 at least one of the two elements b^*b or bb^* is not invertible in B. Suppose b^*b is not invertible in B (the other case proceeds analogously). b^*b is self-adjoint and in particular normal so the version of the proposition already proven applies and $\sigma_A(b^*b) = \sigma_B(b^*b)$. In particular, b^*b is not invertible in A and hence b cannot be invertible in A. This completes the proof. □

4 The GNS construction

We now move towards a characterization of noncommutative C*-algebras. We are going to show that any unital C*-algebra is isomorphic to C*-subalgebra of the bounded operators on some Hilbert space.

Definition 4.1. Let A be a unital C*-algebra. A self-adjoint element $a \in A$ is called *positive* iff $\sigma_A(a) \subset [0, \infty)$.

<u>Exercise</u> 1. Let T be a compact Hausdorff space and consider the C*-algebra $C(T, \mathbb{C})$. Show that the self-adjoint elements are precisely the real valued functions and the positive elements are the functions with non-negative values.

Proposition 4.2. *Let A be a unital C*-algebra and $a, b \in A$ positive. Then, $a+b$ is positive.*

Proof. Suppose $\lambda \in \sigma_A(a+b)$. Since a and b are self-adjoint so is $a+b$. In particular, $\sigma_A(a+b) \subset \mathbb{R}$ and λ is real. Set $\alpha := \|a\|$ and $\beta := \|b\|$. Then, $(\alpha+\beta)-\lambda \in \sigma_A((\alpha+\beta)e-(a+b))$ and thus $|(\alpha+\beta)-\lambda| \le r_A((\alpha+\beta)e-(a+b))$ by Theorem 1.14. But the element $(\alpha+\beta)e-(a+b)$ is normal (and even self-adjoint), so Proposition 3.3 applies and we have $r_A((\alpha+\beta)e-(a+b)) = \|(\alpha+\beta)e-(a+b)\| \le \|\alpha e - a\| + \|\beta e - b\|$. Again using Proposition 3.3 we find $\|\alpha e - a\| = r_A(\alpha e - a)$ and $\|\beta e - b\| = r_A(\beta e - b)$. But $\sigma_A(a) \subseteq [0, \alpha]$ by positivity and Proposition 1.7. Thus, $\sigma_A(\alpha e - a) \subseteq [0, \alpha]$. Hence, by Theorem 1.14, $r_A(\alpha e - a) \le \alpha$. In the same way we find $r_A(\beta e - b) \le \beta$. We have thus demonstrated the inequality $|(\alpha+\beta)-\lambda| \le \alpha+\beta$. This implies $\lambda \ge 0$, completing the proof. □

Proposition 4.3. *Let A be a unital C*-algebra and $a \in A$ self-adjoint. Then, there exist positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.*

Proof. <u>**Exercise.**</u> Hint: Consider the unital C*-subalgebra generated by a. □

Proposition 4.4. *Let A be a unital C*-algebra and $a \in A$. Then, a is positive iff there exists $b \in A$ such that $a = b^* b$.*

Proof. <u>**Exercise.**</u> □

A similar role to that played by the characters in the theory of commutative C*-algebras is now played by *states*.

Definition 4.5. Let A be a unital C*-algebra. A linear functional $\omega : A \to \mathbb{C}$ is called a *state on A* iff $\omega(a) \ge 0$ for all positive elements $a \in A$. A state is called *normalized* iff $\|\omega\| = 1$.

Proposition 4.6. *Let A be a unital C*-algebra and ω a state on A. Then $\omega(a^*) = \overline{\omega(a)}$ for all a in A. In particular, $\omega(a) \in \mathbb{R}$ if a is self-adjoint.*

Proof. <u>**Exercise.**</u> □

Proposition 4.7. *Let A be a unital C*-algebra and ω a state on A. Consider the map $[\cdot,\cdot]_\omega : A \times A \to \mathbb{C}$ given by $[a,b]_\omega = \omega(b^* a)$. It has the following properties:*

1. *$[\cdot,\cdot]_\omega$ is a sesquilinear form on A.*
2. *$[a,b]_\omega = \overline{[b,a]_\omega}$ for all $a, b \in A$.*
3. *$[a,a]_\omega \ge 0$ for all $a \in A$.*

Proof. **Exercise**. □

This shows that we almost have a scalar product, only the definiteness condition is missing. Nevertheless we have the Cauchy-Schwarz inequality.

Proposition 4.8. *Let A be a unital C^*-algebra and ω a state on A. The following is true:*

1. $|[a,b]_\omega|^2 \leq [a,a]_\omega [b,b]_\omega$ *for all* $a, b \in A$.
2. *Let* $a \in A$. *Then,* $[a,a]_\omega = 0$ *iff* $[a,b]_\omega = 0$ *for all* $b \in A$.
3. $[ab, ab]_\omega \leq \|a\|^2 [b,b]_\omega$ *for all* $a, b \in A$.

Proof. **Exercise**. □

Proposition 4.9. *Let A be a unital C^*-algebra and ω a state on A. Define $I_\omega := \{a \in A : [a,a]_\omega = 0\} \subseteq A$. Then, I_ω is a left ideal of the algebra A. In particular, the quotient vector space A/I_ω is a pre-Hilbert space (scalar product space) with the induced sesqulinear form.*

Proof. **Exercise**. □

Definition 4.10. Let A be a unital C*-algebra and ω a state on A. We call the completion of the pre-Hilbert space A/I_ω the *Hilbert space associated with the state* ω and denote it by H_ω. We denote its scalar product by $\langle \cdot, \cdot \rangle_\omega : H_\omega \times H_\omega \to \mathbb{C}$.